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Indentation at a bimaterial interface: the line force solution

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Abstract

The problem considered is that of a line force acting normal to the free surface of a bimaterial welded together at the bimaterial interface. The constitutive equations of each constituent of the bimaterial are such that the stress–strain relation is linear in one half but non-linear in the other. The case where both satisfy non-linear stress–strain relations can also be dealt with by the method of this paper. It is shown that in general, there is no separable solution, singular at the application of the line force, which satisfies the field equations of each medium. Instead asymptotic solutions are constructed for the cases where the non-linear medium is incompressible with power law stress–strain relation (i.e. $\sigma \approx \varepsilon^N$, with $N > 0$) distinguishing between the cases where N is less or greater than unity. The characteristics of the asymptotic solutions are first illustrated by modelling non-linear potential problems, which can be viewed as anti-plane strain deformation. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Bimaterial; Interface; Line force; Power law stress–strain relation; Stress singularities

1. Introduction

We consider the situation depicted in Fig. 1. A bimaterial is constructed of two half-spaces welded together along a common half-plane. The medium is assumed infinite in a direction normal to the figure. Loads are applied to a free surface which is perpendicular to this common half-plane. The direction of the load is perpendicular to the free surface and is a line load acting on the line at which the bimaterial boundary cuts the free surface. Fig. 1 shows a two-dimensional cross-section, all loads and displacements being independent of the spatial coordinate perpendicular to the figure. Bogy (1970) solved this problem for the case of linear elasticity in both materials, and the concentrated force as a particular case of arbitrary boundary tractions. We begin with a description of a model potential problem (anti-plane strain) and then proceed to the case when the bimaterial consists of a linear elastic plus a power law hardening (softening) incompressible material (e.g. a material which deforms according to the theory of deformation plasticity).

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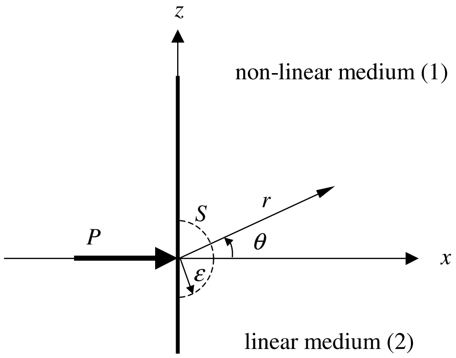

Fig. 1. Cross-section perpendicular to the line load direction: plane strain deformation.

1.1. A model potential problem (anti-plane strain deformation)

It is most convenient to describe these problems in terms of anti-plane strain deformation. The situation is depicted in the two-dimensional cross-section shown in Fig. 2. A line load acts along the y-axis imposing shear in the y-direction on a free surface at $z = 0$.

The region $x > 0$ is occupied by a material such that the only non-zero stresses are

$$\sigma_{xy}^{(1)} = \mu_1 |\nabla\phi_1|^{N-1} \frac{\partial\phi_1}{\partial x}, \qquad \sigma_{zy}^{(1)} = \mu_1 |\nabla\phi_1|^{N-1} \frac{\partial\phi_1}{\partial z}, \tag{1.1}$$

where

$$|\nabla\phi_1|^2 = \left(\frac{\partial\phi_1}{\partial x}\right)^2 + \left(\frac{\partial\phi_1}{\partial z}\right)^2. \tag{1.2}$$

The equilibrium equations

$$\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{zy}}{\partial z} = 0 \tag{1.3}$$

give for the constitutive equations (1.1) the non-linear partial differential equation

$$\frac{\partial}{\partial x}\left(|\nabla\phi_1|^{N-1} \frac{\partial\phi_1}{\partial x}\right) + \frac{\partial}{\partial z}\left(|\nabla\phi_1|^{N-1} \frac{\partial\phi_1}{\partial z}\right) = 0 \quad \text{for } x > 0,\ z > 0. \tag{1.4}$$

In the region $x < 0$, the material is linear with shear modulus, μ_2 and stresses

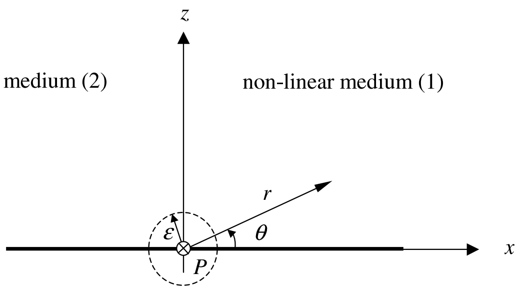

Fig. 2. Cross-section perpendicular to the line load direction: anti-plane strain deformation.

$$\sigma_{xy}^{(2)} = \mu_2 \frac{\partial \phi_2}{\partial x}, \qquad \sigma_{zy}^{(2)} = \mu_2 \frac{\partial \phi_2}{\partial z}. \tag{1.5}$$

The equilibrium equations (1.3) result in

$$\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial z^2} = 0 \quad \text{for } x < 0,\ z > 0. \tag{1.6}$$

The welded boundary conditions on the bimaterial interface require continuity of displacements and tractions across the plane $x = 0$. These can be written as

$$\begin{matrix} \phi_1 = \phi_2 \\ \sigma_{xy}^{(1)} = \sigma_{xy}^{(2)} \end{matrix} \quad \text{on } x = 0,\ z > 0. \tag{1.7}$$

For the situation when medium (1) is linear i.e. when $N = 1$, the well-known solution for a line force (or two-dimensional point force) acting at the origin $x = 0$, $z = 0$ can be written as

$$\phi_1 = \phi_2 = A \ln r, \tag{1.8}$$

where $r^2 = x^2 + z^2$ and the constant A being determined from the strength of the force.

This solution satisfies the continuity conditions (1.7) as well as zero traction on the free surface $z = 0$ except for the load point where the stress is singular. The solution given in Eq. (1.8) is no longer valid in both media when $N \neq 1$ in medium (1). For this situation, the form of the solution differs for the cases where the exponent N is less or greater than one.

If $N < 1$ in region (1), then assuming a solution of the form (1.8) in region (2) leads, by continuity of displacements on the bimaterial interface, to a stress $\sigma_{\theta y}^{(1)}$ in medium (1) which is singular like r^{-N}, where r is a two-dimensional radial coordinate measured from the point of action of the point force. This stress field which is non-zero at the bimaterial interface requires a displacement in region (2) which behaves like r^{1-N} as r tends to zero (cf. constitutive equations (1.5)) in order to satisfy continuity of $\sigma_{\theta y}$ on $x = 0$. Matching this displacement (for displacement continuity on $x = 0$) leads to the next term in the displacement and stress expansion for small r in medium (1). In this way, an expansion of the solution in both media valid for small r is obtained which satisfies zero tractions on the free surface and continuity of both traction and displacement at the bimaterial interface. The details of this solution require solution of various subsidiary boundary value problems and subsequent numerical solution of certain non-linear differential equations (Section 2.1).

If $N > 1$ in medium (1), the nature of the solution is somewhat different. For this case, the leading term in the expansion for the solution for small r consists of the line force (two-dimensional point force) solution for the power law material of a homogeneous medium. This has stress field $\sigma_{ry}^{(1)} = Ar^{-1}$ with corresponding displacement ϕ_1 which behaves like $r^{1-(1/N)}$ as r tends to zero. For continuity of displacements across the bimaterial interface, this requires ϕ_2 behaving like $r^{1-(1/N)}$ also, which results in $\sigma_{ry}^{(2)}$ proportional to $r^{-1/N}$ as r tends to zero. Subsequent terms in the expansion of the solution in each medium are derived in Section 2.2, where the various angular forms of the terms in the solution are also given.

1.2. Plane strain deformation

The situation considered here is shown in Fig. 1. The constitutive relation governing the material in medium (1) is a power law hardening (softening) relation between the stresses σ_{ij} and strains ε_{ij} of the form

$$s_{ij} = 2\mu \Gamma^{N-1} \varepsilon_{ij}, \tag{1.9}$$

where N is the hardening coefficient, the stress deviator is

$$s_{ij} = \sigma_{ij} - \sigma \delta_{ij}, \tag{1.10}$$

where σ is an arbitrary pressure and

$$\Gamma^2 = 2\varepsilon_{ij}\varepsilon_{ij}. \tag{1.11}$$

The summation convention over repeated indices is used here.

$\mu\Gamma^{N-1}$ plays the role of a generalised shear modulus for the physically non-linear material. In the case $N = 1$, it is just the shear modulus of the linear elastic medium. This constitutive relation has been used by Arutiunian and others to study contact problems in non-linear creep (see Arutiunian (1967) for a review and Atkinson (1971)). It has also been used to study plastic stress and strain fields at crack tips (see e.g. Hutchinson (1968a,b), Rice and Rosengren (1968)). The problem of the stress field of crack tips on bimaterial interfaces has been considered by Champion and Atkinson (1990). Lau and Delale (1988) considered the problem of two bonded wedges with power-law hardening, but only for two equal wedges of total angle 180°. Xia and Wang (1993) studied the case of a single wedge of arbitrary angle with various hardening exponents, while the case of two bonded of arbitrary angle and having the same hardening exponent has been studied by Rudge (1993) and Rudge and Tiernan (1999). For this case, since the exponents are the same for each medium, a single separable solution can be looked for which is valid in a region close to the wedge tip resulting in a single non-linear eigenvalue equation.

The nature of the solution for the situation considered here is much more complicated than that of Section 1.1 but the argument is somewhat similar. For the case when the exponent N is less than unity, the leading solution corresponds to a point force acting on a half-space of a homogeneous linear elastic medium. This solution has a displacement field which is singular like $\ln r$ and a stress field singular like r^{-1}. As before, the strength of the force determines this singular field precisely. Continuity of displacements at the bimaterial interface leads to a displacement field in medium (1) which is singular like $\ln r$ and a stress field singular like r^{-N} with appropriate coefficients functions of the angular coordinate which satisfy non-linear ordinary differential equations. This suggests that the next term for the displacements in the linear medium (2) behaves like r^{1-N} times an appropriate function of theta as r tends to zero and so on. The asymptotic expansion of the various fields, valid as r tends to zero, are given in Section 3.2.

For the case $N > 1$, the situation parallels that discussed in Section 1.1 for the anti-plane problem. The leading term in medium (1) is now that of a point force acting on a half-space of a material with constitutive equations (1.9). The solution for the particular case of a point force acting normal to the free surface was given in Arutiunian (1959) for plane strain and in Atkinson (1973) for plane stress. This solution has the stress $\sigma_{ij}^{(1)}$ singular like r^{-1} as r tends to zero and hence a displacement like $r^{1-(1/N)}$. There are of course two non-zero components of displacement here with appropriate angular variation. A solution in the linear medium (2) can be constructed which is traction free on the free surface and is continuous with the displacement above on the bimaterial interface i.e. the displacement behaves like $r^{1-(1/N)}$ as r tends to zero in medium (2). The corresponding stress is then singular like $r^{-1/N}$. Again successive terms in an asymptotic expansion, valid as r tends to zero, can be set up. See Section 3.3 for the details.

In Section 4, a number of results are collected and the various angular variations of the solutions displayed (as above these will result from numerical solutions of ordinary differential equations). Note that the procedure outlined here will apply equally well, at the expense of some complication, to situations where each of the bimaterial media satisfies a different non-linear constitutive relation.

One further point is worthy of mention: we have assumed that the force acting on the interface is also in the plane of the interface (and perpendicular to the free surface for the plane strain situation) and satisfied the balance of force conditions for this to be so. However, because of the asymmetry of the problem, some sort of control procedure might be necessary in practice since applying a line load in a direction normal to the boundary would encounter a net force perpendicular to the interface due to the asymmetry. Either of the above two situations can be modelled by the analysis described below.

As discussed above, we generate the solution as a sum of solutions of subsidiary problems each of which has a factor involving an exponent of r (occasional solutions also involve factors logarithmic in r). These solutions, which are valid for r tending to zero, have exponents which are higher powers of r as the series progresses so that successive solutions decrease in magnitude of r sufficiently small. Since our purpose is to generate the small r solution in a systematic way, we have not posed the problem in a manner that would lead to an existence theorem. We could probably have set the problem up in terms of non-linear integral equations and studied the problem from a theoretical point of view (see e.g. Champion and Atkinson (1993), where this is done for a non-linear bimaterial interface crack).

2. Anti-plane strain deformation: analysis

A two-dimensional cross-section and the polar coordinate system are shown in Fig. 2. The line load acts on the free surface $z = 0$, along the y-axis which is normal to the plane of the paper.

Asymptotic solutions

$$\phi_i = \phi_{i0} + \phi_{i1} + \phi_{i2} + \cdots \tag{2.1}$$

with

$$\lim_{r\to 0} \frac{\phi_{i(j+1)}}{\phi_{ij}} = 0 \quad \forall j \geqslant 0, \;\; i = 1, 2 \tag{2.2}$$

will be constructed for both materials.

2.1. Case $N < 1$

In this case, we take the solution for the homogeneous elastic medium as the leading term of the solution for the medium (1), i.e.

$$\phi_{20} = A_1 \ln r, \tag{2.3}$$

where A_1 is a constant that will be calculated from the strength of the force P.

The continuity of displacements on the interface leads us to a leading term for the non-linear medium (1) of the form

$$\phi_{10} = A_1(\ln r + G_1(\theta)) \tag{2.4}$$

(note that the angular term contributes to the singularity of the stresses).

The equilibrium equation (1.4) then gives the non-linear differential equation

$$(1-N)\left[1 + \left(G_1'(\theta)\right)^2\right]^{(N-1)/2} + \frac{\mathrm{d}}{\mathrm{d}\theta}\left\{\left[1 + \left(G_1'(\theta)\right)^2\right]^{(N-1)/2} G_1'(\theta)\right\} = 0, \tag{2.5}$$

where $G_1'(\theta) = \mathrm{d}G_1(\theta)/\mathrm{d}\theta$.

The boundary conditions of zero traction on the free surface and continuity of displacements on the interface give respectively the conditions

$$G_1'(0) = 0 \quad \text{and} \quad G_1\left(\frac{\pi}{2}\right) = 0. \tag{2.6}$$

Results for Eq. (2.5) subject to the boundary conditions (2.6) are shown in Section 4, in Fig. 3.

Using the constitutive equations (1.1), the solution (2.4) gives a stress field singular like r^{-N} when r tends to zero, which is not zero on the interface. The condition of traction continuity on the interface requires a displacement field in medium (2) which behaves like r^{1-N} when r tends to zero. We get this by adding to the asymptotic expansion of the solution ϕ_2 a new term as follow:

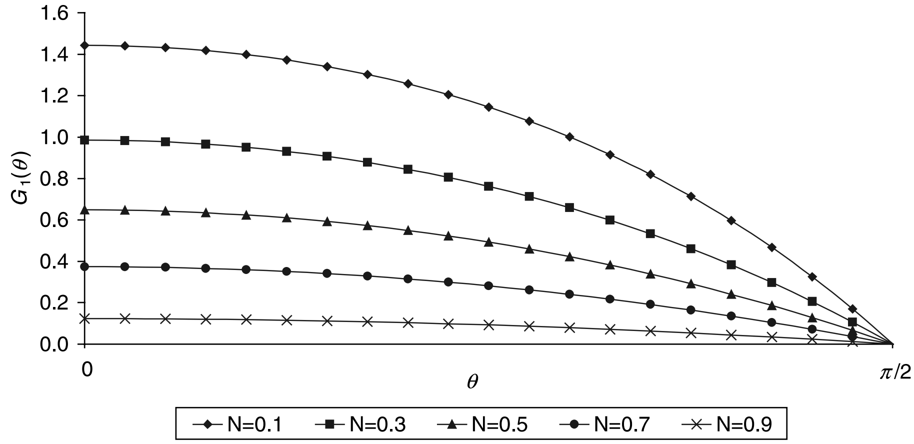

Fig. 3. Angular coefficient $G_1(\theta)$ in the leading term of the displacement of the non-linear medium (anti-plane strain deformation) for different values of $N < 1$.

$$\phi_2 = A_1 \ln r + A_2 r^{1-N} \cos((1-N)(\pi - \theta)). \tag{2.7}$$

This function ϕ_2 satisfies Laplace's equation and the free surface condition. Imposing the continuity of stresses on the interface, we obtain for A_2

$$A_2 = \frac{\mu_1}{\mu_2} \frac{\left[1 + \left(G_1'\left(\frac{\pi}{2}\right)\right)^2\right]^{(N-1)/2} G_1'\left(\frac{\pi}{2}\right)}{(1-N)\sin\left[(1-N)\frac{\pi}{2}\right]} A_1^N. \tag{2.8}$$

Eq. (2.7) shows that, in order to match displacements on the bimaterial interface, the next term of the expansion of the solution ϕ_1 should be r^{1-N} times a function of θ.

We take

$$\phi_1 = A_1\left\{\ln r + G_1(\theta) + r^{1-N} G_2(\theta)\right\}, \tag{2.9}$$

where the function $G_2(\theta)$ should satisfy (cf. Eq. (1.4)) the linear differential equation

$$\begin{aligned}&(1-N)^2\left\{2\alpha(\theta)^{(N-1)/2} - \left[2(1-N) + G_1''(\theta)\right]\alpha(\theta)^{(N-3)/2} - (N-3)\left(G_1'(\theta)\right)^2 G_1''(\theta)\alpha(\theta)^{(N-5)/2}\right\}G_2(\theta)\\ &\quad + (1-N)\left\{3G_1'(\theta)\left[N-1-G_1''(\theta)\right]\alpha(\theta)^{(N-3)/2} - (N-3)\left(G_1'(\theta)\right)^3 G_1''(\theta)\alpha(\theta)^{(N-5)/2}\right\}G_2'(\theta)\\ &\quad + \left\{\alpha(\theta)^{(N-1)/2} + (N-1)\left(G_1'(\theta)\right)^2\alpha(\theta)^{(N-3)/2}\right\}G_2''(\theta) = 0,\end{aligned} \tag{2.10}$$

where $\alpha(\theta) = 1 + \left(G_1'(\theta)\right)^2$.

The free surface condition requires

$$G_2'(0) = 0 \tag{2.11}$$

and the continuity of displacements on the interface

$$A_1 G_2\left(\frac{\pi}{2}\right) = A_2 \cos\left[(1-N)\frac{\pi}{2}\right]. \tag{2.12}$$

The differential equation (2.10) is solved numerically (Section 4).

The solution (2.9) for ϕ_1 adds an additional contribution to the stress that is singular when r tends to zero like r^{1-2N}, which leads to the addition of the next term of the solution ϕ_2

$$\phi_2 = A_1 \ln r + A_2 r^{1-N} \cos((1-N)(\pi-\theta)) + A_3 r^{2(1-N)} \cos(2(1-N)(\pi-\theta)) \tag{2.13}$$

with

$$A_3 = \frac{\mu_1 A_1^{N+1}}{2\mu_2(1-N)\sin\left[2(1-N)\frac{\pi}{2}\right]} \left\{ (N-1)\alpha\left(\frac{\pi}{2}\right)^{(N-3)/2} G_1'\left(\frac{\pi}{2}\right)\left[(1-N)G_2\left(\frac{\pi}{2}\right) + G_1'\left(\frac{\pi}{2}\right)G_2'\left(\frac{\pi}{2}\right)\right] + G_2'\left(\frac{\pi}{2}\right)\alpha\left(\frac{\pi}{2}\right)^{(N-1)/2} \right\} \tag{2.14}$$

in order to satisfy the continuity of stresses on the interface.

Proceeding in the same manner as above, the next terms of the asymptotic expansion of the solutions can be deduced. It remains to calculate the value of the coefficient A_1 in the leading term of both functions ϕ_1 and ϕ_2 (cf. Eqs. (2.3) and (2.4)). The net force acting on the origin is

$$F_i = \int_S \sigma_{ij}\, \mathrm{d}s_j, \quad i = 1,2,3, \tag{2.15}$$

where S is an infinitesimal surface surrounding the origin (cf. Fig. 2). As in this case, the only non-zero stresses are σ_{xy} and σ_{zy}, the components F_x and F_z are zero. The component F_y (that should be equal to P) in cylindrical coordinates can be written as

$$P = \lim_{\varepsilon\to 0} \int_0^{\pi} \sigma_{ry}\, \varepsilon\, \mathrm{d}\theta, \tag{2.16}$$

this implies

$$P = \mu_1 \lim_{\varepsilon\to 0} \int_0^{\pi/2} |\nabla\phi_1|^{N-1} \frac{\partial\phi_1}{\partial r} \varepsilon\, \mathrm{d}\theta + \mu_2 \lim_{\varepsilon\to 0} \int_{\pi/2}^{\pi} \frac{\partial\phi_2}{\partial r} \varepsilon\, \mathrm{d}\theta. \tag{2.17}$$

For the non-linear medium, $|\nabla\phi_1|^{N-1}(\partial\phi_1/\partial r) \approx \varepsilon^{-N}$ (cf. Eq. (2.9)) and in this case $N < 1$, so the first term of the right hand side of Eq. (2.17) becomes zero.

In the integral corresponding to the linear medium

$$\frac{\partial\phi_2}{\partial r}\varepsilon = A_1 + \varepsilon^{1-N}\left[A_2 \cos((1-N)(\pi-\theta)) + A_3 \varepsilon^{1-N} \cos(2(1-N)(\pi-\theta)) + \cdots\right] \tag{2.18}$$

so only the leading term of the expansion of ϕ_2 contributes to the integral and the following value for A_1 is obtained:

$$A_1 = \frac{2P}{\pi\mu_2}. \tag{2.19}$$

2.2. Case $N > 1$

In this case, the line force solution for the power law material of a homogeneous medium is taken as the leading term in the expansion for the solution of the bimaterial

$$\phi_{10} = B_1 r^{1-(1/N)}, \tag{2.20}$$

where B_1 is a constant that will be determined later.

The continuity of displacements on the interface requires for the linear medium (2) a leading term of the form

$$\phi_{20} = B_2 r^{1-(1/N)} \cos\left[\left(1-\frac{1}{N}\right)(\pi-\theta)\right] \tag{2.21}$$

with

$$B_2 = \frac{B_1}{\cos\left[\left(1-\frac{1}{N}\right)\frac{\pi}{2}\right]}. \tag{2.22}$$

This solution for the linear medium satisfies the equilibrium equations and the boundary conditions of free traction on the free surface and continuity of displacements on the interface. However, the stress field associated with these displacements is singular like $r^{-(1/N)}$ when r tends to zero. To match the stress on the interface, the next term of the expansion of the non-linear medium is added.

If we consider a solution of the form

$$\phi_1 = B_1 r^{1-(1/N)} + r^{2(1-(1/N))} f_1(\theta), \tag{2.23}$$

the solution for the equilibrium equation with the boundary conditions of continuity of stresses on the interface and zero traction on the free surface is

$$f_1(\theta) = \frac{-\beta}{\sqrt{2N}\left(1-\frac{1}{N}\right)\sin\left[\sqrt{2N}\left(1-\frac{1}{N}\right)\frac{\pi}{2}\right]} \cos\left[\sqrt{2N}\left(1-\frac{1}{N}\right)\theta\right] \tag{2.24}$$

with

$$\beta = \frac{\mu_2}{\mu_1}\frac{B_2}{B_1^{N-1}}\left(1-\frac{1}{N}\right)^{2-N} \sin\left[\left(1-\frac{1}{N}\right)\frac{\pi}{2}\right]. \tag{2.25}$$

However, this solution is not valid for those values of N that satisfy $\sin[\sqrt{2N}(1-(1/N))\frac{\pi}{2}] = 0$.

In this case, a general function is taken as the second term of the expansion

$$\phi_1 = B_1 r^{1-(1/N)} + F_1(r,\theta). \tag{2.26}$$

In order to satisfy the equilibrium equation (1.4) to the next order, the function $F_1(r,\theta)$ should satisfy

$$N\frac{\partial^2 F_1}{\partial r^2} + \frac{1}{r}\frac{\partial F_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F_1}{\partial \theta^2} = 0. \tag{2.27}$$

The boundary conditions of continuity of stresses on the interface and free tractions on the free surface require the conditions

$$\begin{aligned} &\frac{\partial F_1}{\partial \theta} = \beta r^{2(1-(1/N))} \quad \text{on } \theta = \frac{\pi}{2}, \\ &\frac{\partial F_1}{\partial \theta} = 0 \quad \text{on } \theta = 0. \end{aligned} \tag{2.28}$$

Solving Eq. (2.27) subject to the conditions (2.28) gives $F_1(r,\theta)$ as a function of the form

$$F_1(r,\theta) = r^{2(1-(1/N))}(f_{11}(\theta) + f_{12}(\theta)\ln r) \tag{2.29}$$

with

$$f_{11}(\theta) = \frac{\beta}{\sqrt{2N}\left(1-\frac{1}{N}\right)\frac{\pi}{2}\cos\left(\sqrt{2N}\left(1-\frac{1}{N}\right)\frac{\pi}{2}\right)}\theta\sin\left[\sqrt{2N}\left(1-\frac{1}{N}\right)\theta\right] + C\cos\left[\sqrt{2N}\left(1-\frac{1}{N}\right)\theta\right], \tag{2.30}$$

$$f_{12}(\theta) = \frac{-\beta}{\frac{3}{2}N\left(1-\frac{1}{N}\right)\frac{\pi}{2}\cos\left[\sqrt{2N}\left(1-\frac{1}{N}\right)\frac{\pi}{2}\right]}\cos\left[\sqrt{2N}\left(1-\frac{1}{N}\right)\theta\right], \tag{2.31}$$

where C is an arbitrary constant. Note that in the denominator of Eqs. (2.30) and (2.31) $\cos\left[\sqrt{2N}(1-(1/N))\pi/2\right] = \pm 1$ since N is such that $\sin\left[\sqrt{2N}(1-(1/N))\pi/2\right] = 0$.

Also, the angular coefficients for the second term of the expansion of the displacements on the non-linear medium change sign as θ varies between 0 and $\pi/2$ often as N gets large. However, it may have little effect in the full solution of ϕ_1 since these terms are multiplied by powers of r larger than the leading term and hence are not so significant as r tends to zero.

The next step will be to add another term to the expansion of the displacements in the linear medium in order to match the displacements on the interface. When $\sin\left[\sqrt{2N}(1-(1/N))\pi/2\right] \neq 0$, the solution (2.23) leads us to take

$$\phi_2 = B_2 r^{1-(1/N)}\cos\left[\left(1-\frac{1}{N}\right)(\pi-\theta)\right] + B_3 r^{2(1-(1/N))}\cos\left[2\left(1-\frac{1}{N}\right)(\pi-\theta)\right] \tag{2.32}$$

with

$$B_3 = \frac{f_1\left(\frac{\pi}{2}\right)}{\cos\left[2\left(1-\frac{1}{N}\right)\frac{\pi}{2}\right]}. \tag{2.33}$$

Note that this is not singular when $N = 2$ since $f_1(\pi/2)$ cancels the denominator of Eq. (2.33) in this case.

When $\sin\left[\sqrt{2N}(1-(1/N))\pi/2\right] = 0$,

$$\phi_2 = B_2 r^{1-(1/N)}\cos\left[\left(1-\frac{1}{N}\right)(\pi-\theta)\right] + r^{2(1-(1/N))}(f_{21}(\theta) + f_{22}(\theta)\ln r) \tag{2.34}$$

with

$$\begin{aligned} f_{21}(\theta) = {} & \frac{\beta}{\frac{3}{2}N\left(1-\frac{1}{N}\right)\frac{\pi}{2}\cos\left[2\left(1-\frac{1}{N}\right)\frac{\pi}{2}\right]}(\pi-\theta)\sin\left[2\left(1-\frac{1}{N}\right)(\pi-\theta)\right] \\ & + \left\{C - \beta\frac{\tan\left[2\left(1-\frac{1}{N}\right)\frac{\pi}{2}\right]}{\frac{3}{2}N\left(1-\frac{1}{N}\right)\cos\left[2\left(1-\frac{1}{N}\right)\frac{\pi}{2}\right]}\right\}\cos\left[2\left(1-\frac{1}{N}\right)(\pi-\theta)\right], \end{aligned} \tag{2.35}$$

$$f_{22}(\theta) = \frac{-\beta}{\frac{3}{2}N\left(1-\frac{1}{N}\right)\frac{\pi}{2}\cos\left[2\left(1-\frac{1}{N}\right)\frac{\pi}{2}\right]}\cos\left[2\left(1-\frac{1}{N}\right)(\pi-\theta)\right]. \tag{2.36}$$

Note that $\cos[2(1-(1/N))\pi/2] \neq 0$ in Eqs. (2.35) and (2.36) since $\sin\left[\sqrt{2N}(1-(1/N))\pi/2\right] = 0$, that means $N \neq 2$.

The next terms of the asymptotic expansion of the solutions could be deduced. In this case, imposing the condition $\int_S \sigma_{ry}\,\mathrm{d}s = P$ for an infinitesimal surface surrounding the origin, and proceeding as in the case $N < 1$ gives for B_1:

$$B_1 = \frac{N}{N-1}\left(\frac{2P}{\pi\mu_1}\right)^{1/N}. \tag{2.37}$$

3. Plane strain deformation: analysis

The coordinate system considered in this case is shown in Fig. 1. As above, the line load acts on the y-axis, but the non-linear and incompressible medium (1) is located in the region $x > 0$, $z > 0$ and the linear medium (2) in $x > 0$, $z < 0$, so the bimaterial interface coincides with the x-axis. All fields are assumed independent of the y-coordinate.

3.1. Basic equations

In this section, we use for the stress and displacement fields of the non-linear medium the same notation as Atkinson and Champion (1991). The constitutive equations take the form (cf. Eqs. (1.9) and (1.10))

$$\sigma_{ij}^{(1)} = \sigma\delta_{ij} + 2\mu_1\Gamma^{N-1}\varepsilon_{ij}^{(1)}, \tag{3.1}$$

where σ is an arbitrary pressure and Γ, the effective strain (cf. Eq. (1.11)).

The strains $\varepsilon_{ij}^{(1)}$ are given by

$$\varepsilon_{ij}^{(1)} = \frac{1}{2}\left(\frac{\partial u_i^{(1)}}{\partial x_j} + \frac{\partial u_j^{(1)}}{\partial x_i}\right), \tag{3.2}$$

where $u_i^{(1)}$ are the cartesian components of the displacement field and (x_1, x_2, x_3) is equivalent to (x, y, z). As the material (1) is incompressible, the following condition must be satisfied:

$$\varepsilon_{ii}^{(1)} = 0, \tag{3.3}$$

where the summation over repeated indices is again used.

The effective stress, T, is defined by

$$T = \left(\tfrac{1}{2}s_{ij}s_{ij}\right)^{1/2} \tag{3.4}$$

with the deviatoric stress components s_{ij} defined in Eq. (1.10).

From the constitutive equations (3.1), the effective stress T is related to the effective strain by

$$T = \mu_1\Gamma^N. \tag{3.5}$$

As the situation considered is a plane problem, it is convenient to work in a polar coordinate system (cf. Fig. 1).

In polar coordinates, the strain–displacement equations (3.2) may be written as

$$\begin{aligned} \varepsilon_{rr}^{(1)} &= \frac{\partial u_r^{(1)}}{\partial r}, \\ \varepsilon_{\theta\theta}^{(1)} &= \frac{1}{r}\left(\frac{\partial u_\theta^{(1)}}{\partial\theta} + u_r^{(1)}\right), \\ \varepsilon_{r\theta}^{(1)} &= \frac{1}{2}\left(\frac{\partial u_\theta^{(1)}}{\partial r} - \frac{1}{r}u_\theta^{(1)} + \frac{1}{r}\frac{\partial u_r^{(1)}}{\partial\theta}\right), \end{aligned} \tag{3.6}$$

the incompressibility conditions (3.3) become

$$\varepsilon_{rr}^{(1)} + \varepsilon_{\theta\theta}^{(1)} = 0 \tag{3.7}$$

and the effective strain

$$\Gamma = 2\left(\varepsilon_{rr}^{(1)^2} + \varepsilon_{r\theta}^{(1)^2}\right)^{1/2}. \tag{3.8}$$

Some other relations which follow from the constitutive equations (3.1) are

$$\sigma = \tfrac{1}{2}\left(\sigma_{rr}^{(1)} + \sigma_{\theta\theta}^{(1)}\right), \tag{3.9}$$

$$s_{rr} = \tfrac{1}{2}\left(\sigma_{rr}^{(1)} - \sigma_{\theta\theta}^{(1)}\right) = 2\mu_1 \Gamma^{N-1} \varepsilon_{rr}^{(1)}, \tag{3.10}$$

$$\sigma_{r\theta}^{(1)} = 2\mu_1 \Gamma^{N-1} \varepsilon_{r\theta}^{(1)}. \tag{3.11}$$

In addition to the above equations, the usual equilibrium equations must be satisfied in both media. In polar coordinates, these may be written as

$$\begin{aligned} &\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0, \\ &\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2}{r}\sigma_{r\theta} = 0. \end{aligned} \tag{3.12}$$

For the non-linear medium, using Eqs. (3.9) and (3.10), these equations may be rewritten as

$$\begin{aligned} &\frac{\partial(\sigma + s_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}^{(1)}}{\partial \theta} + \frac{2}{r}s_{rr} = 0, \\ &\frac{\partial \sigma_{r\theta}^{(1)}}{\partial r} + \frac{1}{r}\frac{\partial(\sigma - s_{rr})}{\partial \theta} + \frac{2}{r}\sigma_{r\theta}^{(1)} = 0 \end{aligned} \tag{3.13}$$

and eliminating the pressure, σ,

$$\frac{\partial}{\partial \theta}\left(\frac{\partial s_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}^{(1)}}{\partial \theta} + \frac{2}{r}s_{rr}\right) = \frac{\partial}{\partial r}\left(r\frac{\partial \sigma_{r\theta}^{(1)}}{\partial r} - \frac{\partial s_{rr}}{\partial \theta} + 2\sigma_{r\theta}^{(1)}\right). \tag{3.14}$$

3.2. Case $N < 1$

As in the last section, asymptotic solutions will be constructed for both materials.

$$\begin{aligned} u_r^{(i)} &= u_{r,0}^{(i)} + u_{r,1}^{(i)} + u_{r,2}^{(i)} \cdots \\ u_\theta^{(i)} &= u_{\theta,0}^{(i)} + u_{\theta,1}^{(i)} + u_{\theta,2}^{(i)} \cdots \end{aligned} \quad (i = 1, 2). \tag{3.15}$$

When the hardening coefficient N is less than unity, the leading term taken for the asymptotic expansion in the linear medium is a solution of the form

$$\begin{aligned} u_{r,0}^{(2)} &= A_1\left[\left(\ln r - \frac{1}{\kappa + 1}\right)\omega(\theta) - \frac{\kappa - 1}{\kappa + 1}\theta\,\omega'(\theta)\right], \\ u_{\theta,0}^{(2)} &= A_1\left[\left(\ln r + \frac{1}{\kappa + 1}\right)\omega'(\theta) + \frac{\kappa - 1}{\kappa + 1}\theta\,\omega(\theta)\right] \end{aligned} \tag{3.16}$$

with

$$\omega(\theta) = \cos(\theta - \theta_1), \tag{3.17}$$

which corresponds to the solution of a line load acting on the surface of a semi-infinite homogeneous medium, forming an angle, θ_1, with the normal to the free surface. A_1 is a constant that will be calculated later and $\kappa = 3 - 4v$ for plane strain.

The corresponding stress field is

$$\sigma^{(2)}_{rr,0} = \frac{8\mu_2 A_1}{(\kappa+1)r}\omega(\theta), \qquad \sigma^{(2)}_{r\theta,0} = \sigma^{(2)}_{\theta\theta,0} = 0 \quad (r > 0). \tag{3.18}$$

In order to match displacements on the bimaterial interface, the displacement field corresponding to the non-linear medium (1) should satisfy on $\theta = 0$

$$\begin{aligned} u^{(1)}_{r,0} &= A_1\left(\ln r - \frac{1}{\kappa+1}\right)\cos\theta_1, \\ u^{(1)}_{\theta,0} &= A_1\left(\ln r + \frac{1}{\kappa+1}\right)\sin\theta_1. \end{aligned} \tag{3.19}$$

We consider a displacement field of the general form

$$\begin{aligned} u^{(1)}_{r,0} &= A_1[g_1(\theta)\ln r + g_2(\theta)], \\ u^{(1)}_{\theta,0} &= A_1[h_1(\theta)\ln r + h_2(\theta)]. \end{aligned} \tag{3.20}$$

The following expressions for g_1 and h_1 are found to remove any dependence on $\ln r$ in the strains (Eq. (3.16)) and permit continuity with the $\ln r$ terms in Eq. (3.19) above:

$$\begin{aligned} g_1(\theta) &= \omega(\theta), \\ h_1(\theta) &= \omega'(\theta), \end{aligned} \tag{3.21}$$

while the functions g_2 and h_2 satisfy

$$\begin{aligned} g_2(0) &= \frac{-\cos\theta_1}{(\kappa+1)}, \\ h_2(0) &= \frac{\sin\theta_1}{(\kappa+1)}. \end{aligned} \tag{3.22}$$

Moreover, the incompressibility condition requires

$$g_1(\theta) + g_2(\theta) + h_2'(\theta) = 0 \tag{3.23}$$

so the only unknown function involved in the displacements is h_2, which will be found by imposing the conditions that the stresses associated with this displacement field satisfy the equilibrium equation.

From Eqs. (3.9)–(3.11) and (3.20), we obtain for the stress field

$$\begin{aligned} s_{rr} &= r^{-N}A(\theta), \\ \sigma^{(1)}_{r\theta} &= r^{-N}B(\theta), \end{aligned} \tag{3.24}$$

where

$$A(\theta) = A_1^N 2\mu_1 g_1\left[4(g_1)^2 + \left(h_2 + h_2''\right)^2\right]^{(N-1)/2}, \tag{3.25}$$

$$B(\theta) = -A_1^N \mu_1\left(h_2 + h_2''\right)\left[4(g_1)^2 + \left(h_2 + h_2''\right)^2\right]^{(N-1)/2}. \tag{3.26}$$

Eq. (3.14) becomes

$$2(1-N)A'(\theta)+B''(\theta)+N(2-N)B(\theta)=0. \tag{3.27}$$

The condition of free traction on the surface $\theta=\pi/2$ implies

$$B'\left(\frac{\pi}{2}\right)+2(1-N)A\left(\frac{\pi}{2}\right)=0, \qquad B\left(\frac{\pi}{2}\right)=0. \tag{3.28}$$

The differential equation (3.27) can be rewritten as

$$B(\theta)=-N(2-N)\int_{\theta}^{\pi/2}(t-\theta)B(t)\,\mathrm{d}t+2(1-N)\int_{\theta}^{\pi/2}A(t)\,\mathrm{d}t, \tag{3.29}$$

where the conditions (3.28) have been used.

Finally, substituting the Eqs. (3.21), (3.25) and (3.26) into Eq. (3.29), a non-linear integro-differential equation for h_2 is obtained with the initial conditions (cf. Eqs. (3.22) and (3.23))

$$h_2(0)=\frac{\sin\theta_1}{\kappa+1}, \qquad h_2'(0)=\frac{-\kappa\cos\theta_1}{\kappa+1}. \tag{3.30}$$

This non-linear integro-differential equation will be solved numerically (Section 4).

To clarify the procedure, before going on with the next step, the results obtained until now are summarised. The leading terms found for each asymptotic expansion of both displacement fields are

$$\begin{aligned} u_{r,0}^{(1)} &= A_1\left[(\ln r-1)\omega(\theta)-h_2'(\theta)\right],\\ u_{\theta,0}^{(1)} &= A_1\left[(\ln r)\omega'(\theta)+h_2(\theta)\right] \end{aligned} \tag{3.31}$$

with $\omega(\theta)=\cos(\theta-\theta_1)$ for the non-linear medium and Eq. (3.16) for linear medium.

As has been shown (cf. Eq. (3.24)), the stresses for medium (1) are singular with singularity like r^{-N} when r tends to zero. This suggests that the next term for the displacements in the linear medium behaves like r^{1-N}. This can be found analytically by the complex representation method (cf. Muskhelishvili (1953)). The stress and displacement fields are written in terms of two analytic functions of the complex variable $z=r\mathrm{e}^{\mathrm{i}\theta}$ as follows

$$\begin{aligned} \sigma_{rr}+\sigma_{\theta\theta} &= 2\left[\varphi'(z)+\overline{\varphi'(z)}\right],\\ \sigma_{rr}-\sigma_{\theta\theta}+2\mathrm{i}\sigma_{r\theta} &= -2\left[\bar{z}\overline{\varphi''(z)}+\frac{\bar{z}}{z}\overline{\psi'(z)}\right],\\ 2\mu(u_r+\mathrm{i}u_\theta) &= \mathrm{e}^{-\mathrm{i}\theta}\left[\kappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}\right], \end{aligned} \tag{3.32}$$

where the bar denotes the complex conjugate.

As a displacement field with singularity like r^{1-N} is expected, the functions

$$\varphi(z)=\alpha z^{1-N}, \qquad \overline{\psi(z)}=\beta\bar{z}^{1-N} \tag{3.33}$$

are taken where α and β are complex constants.

Imposing the boundary conditions

$$\begin{aligned} &\sigma_{\theta\theta}^{(2)}=\sigma_{r\theta}^{(2)}=0 \quad \text{on } \theta=-\frac{\pi}{2},\\ &\sigma_{\theta\theta}^{(1)}=\sigma_{\theta\theta}^{(2)}, \qquad \sigma_{r\theta}^{(1)}=\sigma_{r\theta}^{(2)} \quad \text{on } \theta=0, \end{aligned} \tag{3.34}$$

the following values for the coefficients $\alpha=\alpha_1+\mathrm{i}\alpha_2$ and β are found (note that in the stresses $\sigma_{r\theta}^{(2)}$ and $\sigma_{\theta\theta}^{(2)}$, there is no contribution from the leading term of the displacements, cf. Eq. (3.18))

$$\alpha_1 = \frac{1}{\Delta(1-N)}\left\{\frac{1}{N}\left[2(1-N)A(0)+B'(0)\right][1-2N-\cos(\pi N)]+B(0)\sin(\pi N)\right\}, \tag{3.35}$$

$$\alpha_2 = \frac{1}{\Delta(1-N)}\left\{\frac{1}{N}\left[2(1-N)A(0)+B'(0)\right]\sin(\pi N)+B(0)[3-2N-\cos(\pi N)]\right\} \tag{3.36}$$

with

$$\Delta = (3-2N)(1-2N)-2\cos(\pi N)-1, \tag{3.37}$$

and

$$\beta = \alpha e^{i\pi N}+(1-N)\overline{\alpha}. \tag{3.38}$$

So the asymptotic expansions for the displacements in medium (2) are (cf. Eqs. (3.16) and (3.32))

$$\begin{aligned} u_r^{(2)} &= A_1\left[\left(\ln r-\frac{1}{\kappa+1}\right)\omega(\theta)-\frac{\kappa-1}{\kappa+1}\theta\,\omega'(\theta)\right]+r^{1-N}\Phi_r^{(2)}(\theta),\\ u_\theta^{(2)} &= A_1\left[\left(\ln r+\frac{1}{\kappa+1}\right)\omega'(\theta)+\frac{\kappa-1}{\kappa+1}\theta\,\omega(\theta)\right]+r^{1-N}\Phi_\theta^{(2)}(\theta), \end{aligned} \tag{3.39}$$

where

$$\Phi_r^{(2)}(\theta)+i\Phi_\theta^{(2)}(\theta) = \frac{1}{2\mu_2}\left[\kappa\alpha e^{-iN\theta}-(1-N)\overline{\alpha}e^{-iN\theta}-\beta e^{i(N-2)\theta}\right]. \tag{3.40}$$

The next step would be to add a suitable term in the asymptotic expansion for the displacements in the non-linear medium that matches with the displacements in medium (2) on the interface. Eq. (3.39) suggests a solution in medium (1) of the form

$$\begin{aligned} u_r^{(1)} &= A_1\left[(\ln r-1)\omega(\theta)-h_2'(\theta)\right]+r^{1-N}\Phi_r^{(1)}(\theta),\\ u_\theta^{(1)} &= A_1\left[(\ln r)\omega'(\theta)+h_2(\theta)\right]+r^{1-N}\Phi_\theta^{(1)}(\theta). \end{aligned} \tag{3.41}$$

The angular coefficients $\Phi_r^{(1)}$ and $\Phi_\theta^{(1)}$ can be found by a similar procedure to that above.

In this case, the only non-zero contribution for the net force acting on the material comes from the leading term of the linear medium. Taking a small contour around the point of the application (cf. Fig. 1), the components of the net force are

$$\begin{aligned} F_x &= \lim_{\varepsilon\to 0}\int_{-\pi/2}^{0}\sigma_{rr}^{(2)}\cos\theta\,\varepsilon\,d\theta = A_1\frac{2\mu_2}{\kappa+1}(-2\sin\theta_1+\pi\cos\theta_1),\\ F_y &= 0,\\ F_z &= \lim_{\varepsilon\to 0}\int_{-\pi/2}^{0}\sigma_{rr}^{(2)}\sin\theta\,\varepsilon\,d\theta = A_1\frac{2\mu_2}{\kappa+1}(-2\cos\theta_1+\pi\sin\theta_1). \end{aligned} \tag{3.42}$$

Imposing the boundary condition of a line force of strength P, i.e. $F_x = P$ and $F_z = 0$, the constants θ_1 and A_1 are

$$\theta_1 = \arctan\frac{2}{\pi}, \qquad A_1 = P\frac{\sqrt{\pi^2+4}(\kappa+1)}{2\mu_2(\pi^2-4)}. \tag{3.43}$$

3.3. Case $N > 1$

The solution for a line load acting on a half space of a power-law elastic material under conditions of plane strain is of the form (Atkinson, 1973)

$$\begin{cases} u_r = C\eta'(\theta)r^{1-(1/N)} + Dv(\theta), \\ u_\theta = C\left(\frac{1}{N} - 2\right)\eta(\theta)r^{1-(1/N)} + Dv'(\theta), \end{cases} \tag{3.44}$$

$$\begin{cases} \sigma_{rr} = 2^{N+1}\mu_1\left(1 - \frac{1}{N}\right)^N C\eta'(\theta)\left|C\eta'(\theta)\right|^{N-1}\frac{1}{r}, \\ \sigma_{yy} = \frac{1}{2}\sigma_{rr}, \\ \sigma_{r\theta} = \sigma_{ry} = \sigma_{\theta\theta} = \sigma_{y\theta} = 0, \end{cases} \tag{3.45}$$

where C is a constant that will be calculated later and D corresponds to a rigid body displacement. The functions $\eta(\theta)$ and $v(\theta)$ are

$$\begin{aligned} \eta(\theta) &= \cos(a(\theta - \theta_1)), \\ v(\theta) &= \cos(\theta - \theta_2) \end{aligned} \tag{3.46}$$

with

$$a = \sqrt{\frac{1}{N}\left(2 - \frac{1}{N}\right)} \tag{3.47}$$

and θ_1, θ_2 constants.

In the case $N > 1$, the above solution is taken as leading term of the asymptotic solution for the non-linear material. Therefore, the leading term of the displacement field for the linear medium should be like $r^{1-(1/N)}$ when r tends to zero. This can be found by means of the complex representation method (cf. Eq. (3.32)). If we suppose that the solution is a function of separable variables, we take

$$\varphi(z) = 2\mu_2\left(\alpha z^{1-(1/N)} + \frac{1}{\kappa + 1}De^{i\theta_2}\right), \qquad \psi(z) = 2\mu_2\left(\beta z^{1-(1/N)} - \frac{1}{\kappa + 1}De^{-i\theta_2}\right) \tag{3.48}$$

with α and β complex constants.

Imposing the conditions of continuity of displacements on the interface the following values for $\alpha = \alpha_1 + i\alpha_2$ are found:

$$\alpha_1 = \frac{C}{\Delta(N)}\left\{a\sin(a\theta_1)\left[\kappa - \cos\left(\frac{\pi}{N}\right) + 2\left(1 - \frac{1}{N}\right)\right] - \cos(a\theta_1)\left(\frac{1}{N} - 2\right)\sin\left(\frac{\pi}{N}\right)\right\} \tag{3.49}$$

$$\alpha_2 = \frac{C}{\Delta(N)}\left\{\cos(a\theta_1)\left(\frac{1}{N} - 2\right)\left[\kappa - 2\left(1 - \frac{1}{N}\right) - \cos\left(\frac{\pi}{N}\right)\right] + a\sin(a\theta_1)\sin\left(\frac{\pi}{N}\right)\right\} \tag{3.50}$$

with

$$\Delta(N) = \kappa^2 - 2\kappa\cos\left(\frac{\pi}{N}\right) + 1 - 4\left(1 - \frac{1}{N}\right)^2. \tag{3.51}$$

The condition of zero traction on the free surface gives

$$\beta = \overline{\alpha}e^{-i\pi/N} + \alpha\left(1 - \frac{1}{N}\right). \tag{3.52}$$

However, this solution is not valid when N satisfies $\Delta(N) = 0$.

In this case, a solution of the form

$$\varphi(z) = z^{1-(1/N)}(\delta^* \log z + \alpha^*) + \frac{2\mu_2}{\kappa + 1} e^{i\theta_2}, \qquad \psi(z) = z^{1-(1/N)}(\chi^* \log z + \beta^*) - \frac{2\mu_2}{\kappa + 1} e^{-i\theta_2} \tag{3.53}$$

has been found. The Appendix A contains the values of the constants α^*, β^*, χ^* and δ^*. In this case, it is only the leading term in the non-linear medium which contributes to the net force acting on the bimaterial. Taking a small contour around the point of the application (cf. Fig. 1), the components of the net force are

$$\begin{aligned} F_x &= \lim_{\varepsilon \to 0} \int_0^{\pi/2} \sigma_{rr}^{(1)} \cos\theta (\varepsilon \, d\theta) = -2^{N+1} \mu_1 \left(1 - \frac{1}{N}\right)^N C \int_0^{\pi/2} \left|C\eta'(\theta)\right|^{N-1} \eta'(\theta) \cos\theta \, d\theta, \\ F_y &= 0, \\ F_z &= \lim_{\varepsilon \to 0} \int_0^{\pi/2} \sigma_{rr}^{(1)} \sin\theta (\varepsilon \, d\theta) = -2^{N+1} \mu_1 \left(1 - \frac{1}{N}\right)^N C \int_0^{\pi/2} \left|C\eta'(\theta)\right|^{N-1} \eta'(\theta) \sin\theta \, d\theta. \end{aligned} \tag{3.54}$$

Imposing the boundary condition of a line force of strength P, the following system of equations with unknowns θ_1 and C is obtained and can be solved numerically.

$$P = -2^{N+1} \mu_1 \left(1 - \frac{1}{N}\right)^N C a^N \int_0^{\pi/2} |C \sin(a(\theta - \theta_1))|^{N-1} \sin(a(\theta - \theta_1)) \cos\theta \, d\theta, \tag{3.55a}$$

$$0 = \int_0^{\pi/2} |\sin(a(\theta - \theta_1))|^{N-1} \sin(a(\theta - \theta_1)) \sin\theta \, d\theta. \tag{3.55b}$$

4. Numerical results and concluding remarks

For the anti-plane strain deformation case when $N < 1$, the expressions for the leading term of the displacements in the two media are

$$\phi_1 = A_1 \{ \ln r + G_1(\theta) + r^{1-N} G_2(\theta) \} + o(r^{1-N}) \qquad \left(0 < \theta < \frac{\pi}{2}\right), \tag{4.1a}$$

$$\phi_2 = A_1 \ln r + A_2 r^{1-N} \cos((1 - N)(\pi - \theta)) + o(r^{1-N}) \qquad \left(\frac{\pi}{2} < \theta < \pi\right), \tag{4.1b}$$

where

$$A_1 = \frac{2P}{\pi \mu_2} \tag{4.2}$$

and P is the strength of the line force and

$$A_2 = \frac{\mu_1}{\mu_2} \frac{\left[1 + \left(G_1'\left(\frac{\pi}{2}\right)\right)^2\right]^{(N-1)/2} G_1'\left(\frac{\pi}{2}\right)}{(1 - N) \sin\left[(1 - N)\frac{\pi}{2}\right]} A_1^N. \tag{4.3}$$

More details of these formulae are given in Section 2.1.

The differential equation (2.5) subject to the condition (2.6) gives the angular coefficient $G_1(\theta)$ corresponding to the leading term of the displacement in the non-linear medium. Using the relation

$$\frac{d}{d\theta} = \frac{dG_1}{d\theta} \frac{dG_1'}{dG_1} \frac{d}{dG_1'}, \tag{4.4}$$

Eq. (2.5) becomes

$$2(1-N)G_1(\theta)+N\big(G_1'(\theta)\big)^2+(1-N)\ln\Big[1+\big(G_1'(\theta)\big)^2\Big]=\text{constant}. \tag{4.5}$$

By differentiating this differential equation several times and using the boundary conditions, the successive derivatives of $G_1(\theta)$ evaluated on $\theta=0$ are calculated to obtain the Taylor expansion of $G_1(\theta)$. In Fig. 3, the function $G_1(\theta)$ is plotted for different values of the hardening coefficient, N.

The angular coefficient $G_2(\theta)$ of the next term should satisfy Eq. (2.10) and the conditions (2.11) and (2.12). This differential equation has been solved numerically with the Runge–Kutta fourth order method. Fig. 4 shows this function for various values of $N<1$. The values of the constant $A_1=1$ unit of length and the ratio $\mu_1/\mu_2=1.5$ have been used (cf. Eqs. (4.1a) and (4.1b)).

It can be observed that, for fixed values of θ, both functions G_1 and G_2 decrease with increasing values of N. For the corresponding plane strain case when $N<1$ (Section 3.2.), the leading terms for the displacement in the solution for the non-linear medium have the form

$$\begin{aligned}u_{r,0}^{(1)}&=A_1[g_1(\theta)\ln r+g_2(\theta)],\\ u_{\theta,0}^{(1)}&=A_1[h_1(\theta)\ln r+h_2(\theta)],\end{aligned} \tag{4.6}$$

where

$$g_1(\theta)=\cos(\theta-\theta_1), \tag{4.7}$$

$$h_1(\theta)=-\sin(\theta-\theta_1) \tag{4.8}$$

and

$$g_2(\theta)=-g_1(\theta)-h_2'(\theta). \tag{4.9}$$

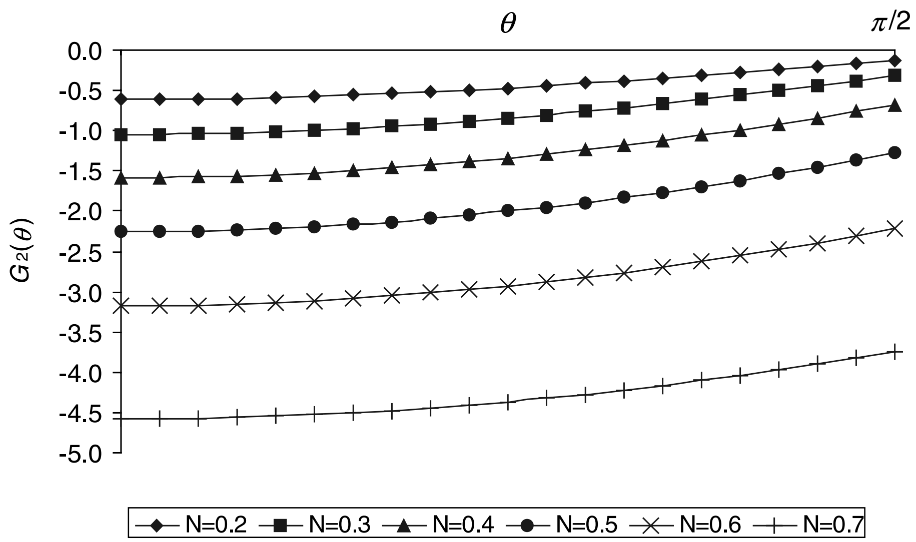

Fig. 4. Angular coefficient $G_2(\theta)$ of r^{1-N} for the displacement of the non-linear medium (anti-plane strain deformation) for different values of $N<1$.

The leading solution in the linear medium has the form (3.16)

$$\begin{aligned} u^{(2)}_{r,0} &= A_1\left[\left(\ln r - \frac{1}{\kappa+1}\right)\cos(\theta-\theta_1) + \frac{\kappa-1}{\kappa+1}\theta\sin(\theta-\theta_1)\right],\\ u^{(2)}_{\theta,0} &= A_1\left[-\left(\ln r + \frac{1}{\kappa+1}\right)\sin(\theta-\theta_1) + \frac{\kappa-1}{\kappa+1}\theta\cos(\theta-\theta_1)\right]. \end{aligned} \tag{4.10}$$

Substituting Eqs. (3.25) and (3.26) in Eq. (3.29) and applying the trapezoidal rule to the definite integrals in the resulting equation, a differential equation for $h_2(\theta)$ is obtained, which is solved numerically using the finite difference method. By dividing $\pi/2$ into the same intervals used in the trapezoidal rule, the differential equation is discretised, obtaining a system of equations that, with the equation supplied by the boundary conditions, is solved by Newton's method. Fig. 5 shows the function h_2 for different values of $N < 1$, where the values $A_1 = 1$ cm, $\kappa = 1.8$ and $\mu_1 = 7.5 \times 10^6$ Kp/cm^2 have been taken. The leading terms of the displacements and normalised stresses $\overline{\sigma}_{ij} = \sigma_{ij}/r^{-N}$ are illustrated in Figs. 6 and 7, respectively.

When $N > 1$ (Section 3.3), the leading terms for the displacements in the solution for the non-linear medium are

$$\begin{aligned} u_r &= C\eta'(\theta)r^{1-(1/N)},\\ u_\theta &= C\left(\frac{1}{N}-2\right)\eta(\theta)r^{1-(1/N)}, \end{aligned} \tag{4.11}$$

where

$$\eta(\theta) = \cos(a(\theta-\theta_1)), \tag{4.12}$$

$$a = \sqrt{\frac{1}{N}\left(2-\frac{1}{N}\right)} \tag{4.13}$$

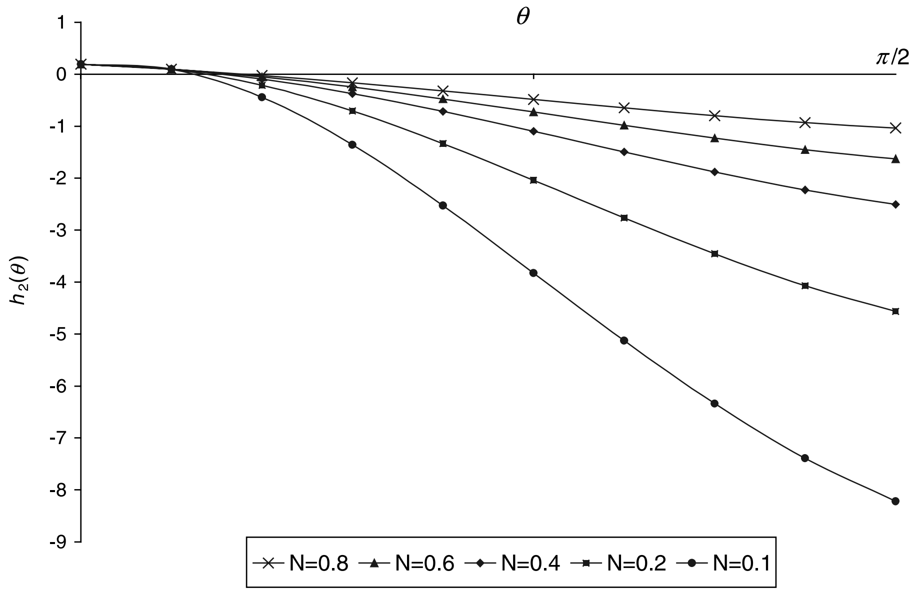

Fig. 5. Angular coefficient $h_2(\theta)$ in leading term of the displacement $u^{(1)}_\theta$ of the non-linear medium (plane strain deformation) for different values of $N < 1$ (cf. Eq. (3.31)).

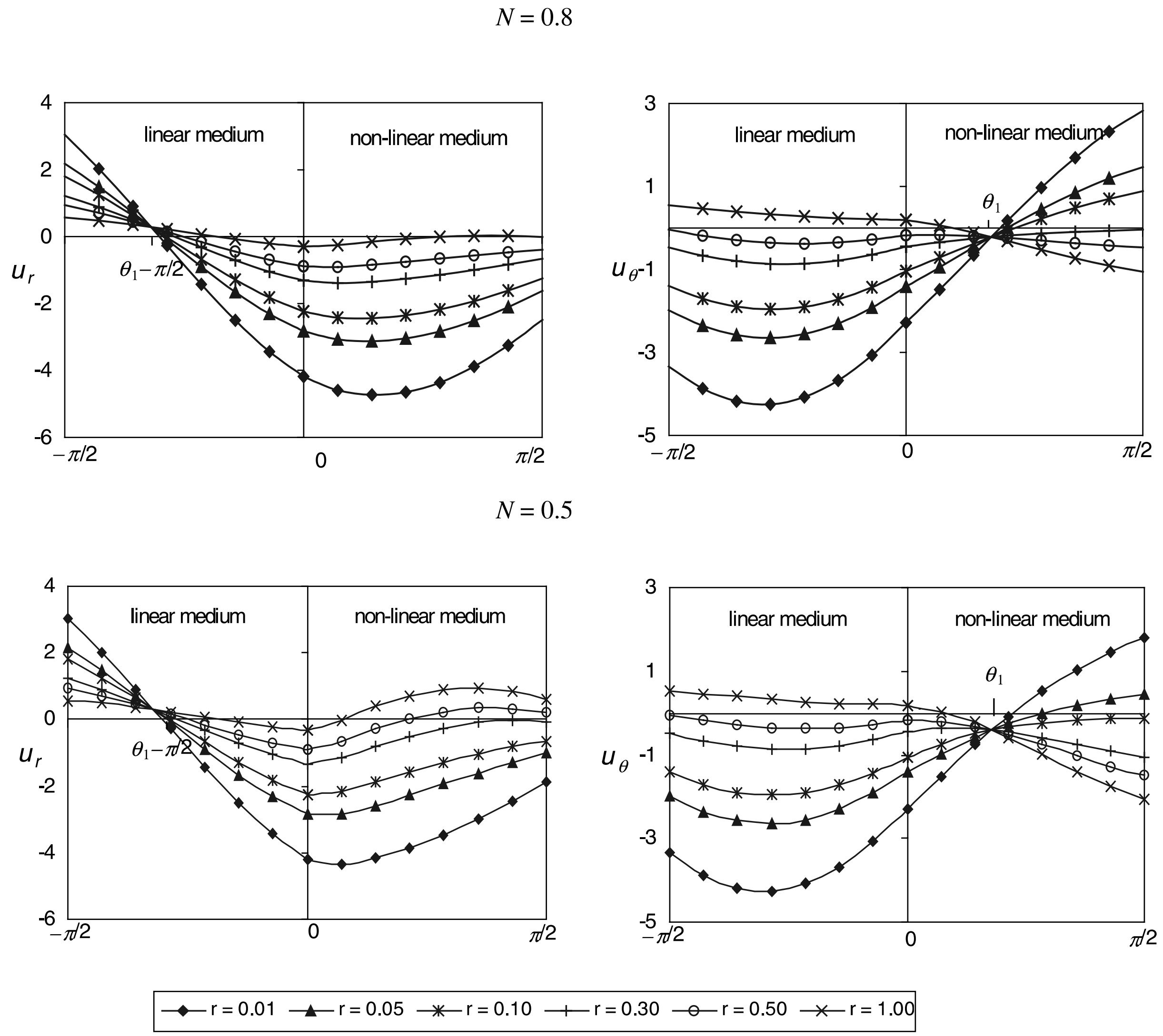

Fig. 6. Leading term of the displacements against θ for various values of $N < 1$ (plane strain deformation).

and the constants C and θ_1 are determined by solving Eqs. (3.55a) and (3.55b). Eq. (3.55b) has been solved with the Bisection Method. Table 1 shows some of the results obtained for both constants for various values of N when $P/\mu_1 = 1$ unit of length.

The expression for leading term of the displacements in the linear medium has the form

$$u_r + \mathrm{i}u_\theta = r^{1-(1/N)}\mathrm{e}^{-\mathrm{i}\theta}\left[\kappa\alpha\mathrm{e}^{\mathrm{i}(1-(1/N))\theta} - \left(1 - \frac{1}{N}\right)\overline{\alpha}\mathrm{e}^{\mathrm{i}(1+(1/N))\theta} - \overline{\beta}\mathrm{e}^{-\mathrm{i}(1-(1/N))\theta}\right]. \tag{4.14}$$

The leading terms of the normalised displacements respect $r^{1-(1/N)}$ for various values of $N > 1$ when $P/\mu_1 = 1$ unit of length and $\kappa = 1.8$ are shown in Fig. 8. The cause of the change of sign in the displacements in the linear material between $N = 3.4$ and $N = 3.5$ is the change of sign of the coefficient $\Delta(N)$ between these values of N (cf. Eq. (3.51)).

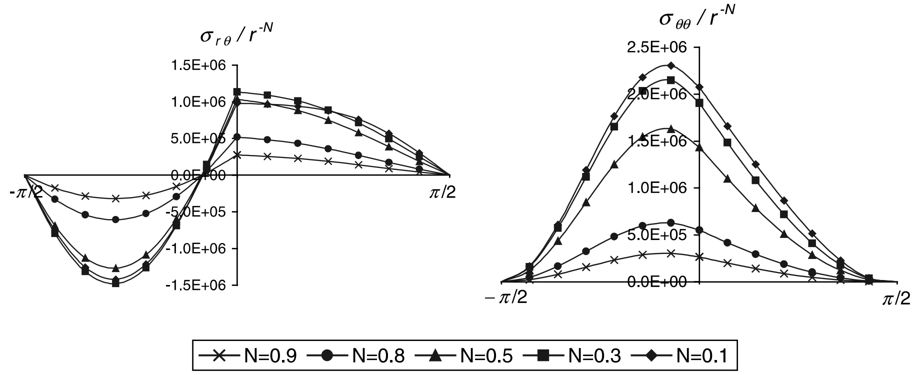

Fig. 7. Leading term of the stresses against θ for various values of $N < 1$ (plane strain deformation).

Table 1
Values for angle, θ_1 and constant, C in leading term of the displacements in the non-linear material (plane strain deformation with $N > 1$, cf. Eqs. (3.55a) and (3.55b)).

N	θ_1	C
1.1	1.0022	7.1967
1.2	0.9990	4.0925
1.3	0.9960	3.0872
1.4	0.9930	2.6053
1.5	0.9901	2.3320
1.6	0.9872	2.1625
1.7	0.9845	2.0522
1.8	0.9818	1.9786
1.9	0.9792	1.9294
2.0	0.9767	1.8973
3.0	0.9553	1.9429
4.0	0.9393	2.1813
5.0	0.9268	2.4631
6.0	0.9168	2.7598
7.0	0.9086	3.0625
8.0	0.9019	3.3675
9.0	0.8962	3.6734
10.0	0.8914	3.9792
15.0	0.8754	5.4999
20.0	0.8664	7.0053

Appendix A

The leading term of the displacement field corresponding to the linear medium for the case of plane strain deformation when $N > 1$ and $\Delta(N) = 0$ is (cf. Eq. (3.53))

$$\varphi(z) = z^{1-(1/N)}\left(\frac{a_1 b_0 - a_0 b_1}{b_0^2} - \frac{a_0}{b_0}\log z\right) + \frac{2\mu_2}{\kappa+1}\mathrm{e}^{\mathrm{i}\theta_2}, \tag{A.1}$$

$$\psi(z) = z^{1-(1/N)}\left(\frac{c_1 b_0 - c_0 b_1}{b_0^2} - \frac{c_0}{b_0}\log z\right) - \frac{2\mu_2}{\kappa+1}\mathrm{e}^{-\mathrm{i}\theta_2} \tag{A.2}$$

with

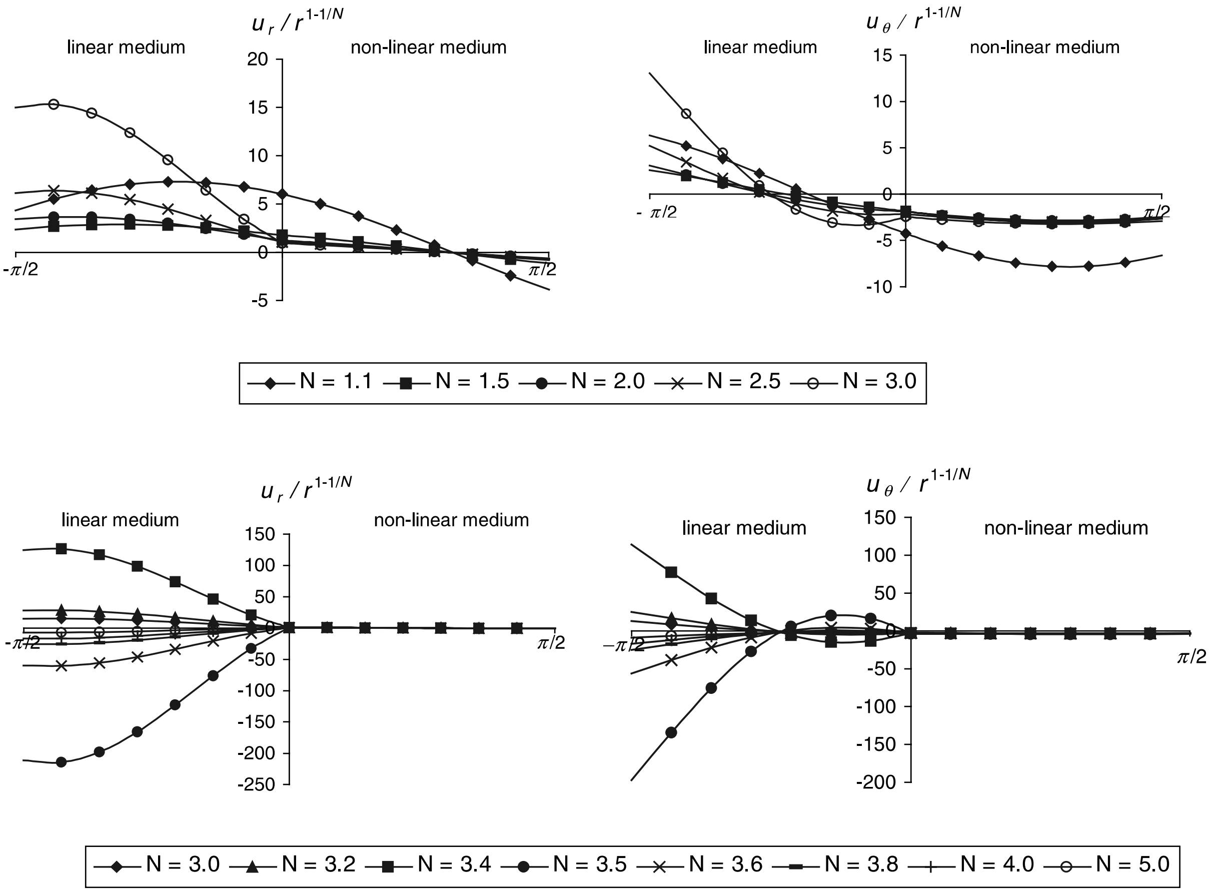

Fig. 8. Leading term of the normalised displacements against θ for various values of $N > 1$ (plane strain deformation).

$$a_0 = U\left(\kappa - \mathrm{e}^{-\mathrm{i}\pi/N}\right) + 2\overline{U}\left(1 - \frac{1}{N}\right), \tag{A.3}$$

$$a_1 = \mathrm{i}\pi U \mathrm{e}^{-\mathrm{i}\pi/N} - 2\overline{U}, \tag{A.4}$$

$$b_0 = 2\kappa\pi \sin\left(\frac{\pi}{N}\right) + 8\left(1 - \frac{1}{N}\right), \tag{A.5}$$

$$b_1 = \kappa\pi^2 \cos\left(\frac{\pi}{N}\right) - 4, \tag{A.6}$$

$$c_0 = U\left(1 - \frac{1}{N}\right)\left(\kappa + \mathrm{e}^{-\mathrm{i}\pi/N}\right) + \overline{U}\left[\kappa \mathrm{e}^{-\mathrm{i}\pi/N} + 2\left(1 - \frac{1}{N}\right)^2 - 1\right], \tag{A.7}$$

$$c_1 = -U\left\{\mathrm{e}^{-\mathrm{i}\pi/N}\left[1 + \mathrm{i}\pi\left(1 - \frac{1}{N}\right)\right] + \kappa\right\} - \overline{U}\left[\mathrm{i}\pi\kappa \mathrm{e}^{-\mathrm{i}\pi/N} + 4\left(1 - \frac{1}{N}\right)\right] \tag{A.8}$$

and the constant U is related with the displacements on the interface (cf. Eq. (3.44))

$$r^{1-(1/N)}U = 2\mu_2\left(u_r^{(1)}(\theta = 0) + \mathrm{i}u_\theta^{(1)}(\theta = 0)\right). \tag{A.9}$$

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